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A theory of dislocations and disclinations in elastic plates☆

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ABSTRACT

The problem of the stress state of a thin elastic plate, containing dislocations and disclinations, is considered using Kirchhoff's theory. The problem of the equilibrium of a multiply connected plate with Volterra dislocations with specified characteristics is formulated. The problem of the flexure of an annular slab resulting from a screw dislocation and a twisting disclination is solved. The solutions of problems of concentrated (isolated) dislocations and disclinations in an unbounded plate as well as the dipoles of dislocations and disinclinations are found. It is shown that a screw dislocation in a thin plate is equivalent to the superposition of two orthogonal dipoles of torsional disclinations. By taking the limit from a discrete set of defects to their continuous distribution, a theory of thin plates with distributed dislocations and disclinations is constructed. Solutions of problems of the flexure of circular and elliptic plates with continuously distributed disclinations are obtained. An analogy is established between the problem of the flexure of a plate with defects and the plane problem of the theory of elasticity with mass forces, and also between a plane problem with dislocations and disclinations and the problem of the flexure of a plate with specified distributed loads.

Defects of the dislocation and disclination type play an important role in the mechanical behaviour of surface crystals, thin-film nanostructures, biological membranes and other two-dimensional physical systems.[1–3]

1. Volterra dislocations in a multiply connected plate

In Kirchhoff's theory,[4] the complete system of equilibrium equations of a linearly elastic plate (slab) consists of the equilibrium equations for the stress and moment tensors (1.1), the constitutive relations (1.2) and the geometrical relations (1.3), namely

$$\nabla\cdot\mathbf{T}+\mathbf{f}=0,\quad \nabla\cdot(\nabla\cdot\mathbf{M})+\nabla\cdot\mathbf{e}\cdot\boldsymbol{\mu}+p=0$$
$$\nabla=\mathbf{i}_1\frac{\partial}{\partial x_1}+\mathbf{i}_2\frac{\partial}{\partial x_2},\quad \mathbf{e}=\mathbf{i}_1\otimes\mathbf{i}_2-\mathbf{i}_2\otimes\mathbf{i}_1 \tag{1.1}$$

$$\mathbf{T}=\frac{\partial W(\boldsymbol{\varepsilon},\mathbf{K})}{\partial\boldsymbol{\varepsilon}},\quad \mathbf{M}=-\frac{\partial W(\boldsymbol{\varepsilon},\mathbf{K})}{\partial\mathbf{K}} \tag{1.2}$$

$$\boldsymbol{\varepsilon}=\frac{1}{2}(\nabla\mathbf{u}+(\nabla\mathbf{u})^T),\quad \mathbf{K}=\nabla\nabla w \tag{1.3}$$

Here, ∇ is the two-dimensional nabla-operator, $\mathbf{e}$ is the discriminant tensor, x_1 and x_2 are Cartesian coordinates in the middle plane of the plate, $\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3=\mathbf{i}_1\times\mathbf{i}_2$ are the coordinate unit vectors, $\mathbf{T}$ and $\mathbf{M}$ are the surface (two-dimensional) stress and moment tensors, W is the specific (per unit area) energy of the plate, $\mathbf{u}$ is the displacement vector field in the plane of the plate, w is the normal deflection, $\boldsymbol{\varepsilon}$ is the plane strain tensor, $\mathbf{K}$ is the flexural deformation tensor, $\mathbf{f}$ and $\boldsymbol{\mu}$ are the plane vectors for the intensities of the force load and the moment load, distributed over the middle plane, and p is the intensity of the transverse load. In Eqs (1.1), we have taken into account the fact that

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a theory of plates and shells of the Kirchhoff type allows of the application of moment load over the middle surface, with a vector which does not have a component directed along the normal to the surface.[5]

If the plate material is isotropic, the specific energy and stress and moment tensors have the form

$$W = \frac{Eh}{2(1-\nu^2)}[(1-\nu)\mathrm{tr}\boldsymbol{\varepsilon}^2 + \nu\mathrm{tr}^2\boldsymbol{\varepsilon}] + \frac{D}{2}[(1-\nu)\mathrm{tr}\mathbf{K}^2 + \nu\mathrm{tr}^2\mathbf{K}]$$
$$\mathbf{T} = \frac{Eh}{1-\nu^2}[(1-\nu)\boldsymbol{\varepsilon} + \nu\mathbf{g}\mathrm{tr}\boldsymbol{\varepsilon}], \quad \mathbf{g} = \mathbf{i}_1\otimes\mathbf{i}_1 + \mathbf{i}_2\otimes\mathbf{i}_2$$
$$\mathbf{M} = -D[(1-\nu)\mathbf{K} + \nu\mathbf{g}\mathrm{tr}\mathbf{K}], \quad D = \frac{Eh^3}{12(1-\nu^2)} \tag{1.4}$$

Here E is Young's modulus, ν is Poisson's ratio, h is the thickness of the plate, D is its cylindrical stiffness and $\mathbf{g}$ is the unit surface tensor. Besides the vector field for the displacements of the middle plane $v = \mathbf{u} + w\mathbf{i}_3$, we shall consider the field of small rotations $\boldsymbol{\Omega}$, defined by the relation

$$\boldsymbol{\Omega} = \mathbf{e}\cdot\boldsymbol{\theta} + \omega\mathbf{i}_3; \quad \boldsymbol{\theta} = \nabla w, \quad \omega = \frac{1}{2}(\nabla\cdot\mathbf{e}\cdot\mathbf{u}) \tag{1.5}$$

where ω is the angle of rotation about the normal to the plate $\mathbf{i}_3$. If the plate moves as an absolutely rigid body, the displacement field has the form

$$\boldsymbol{\upsilon} = \boldsymbol{\upsilon}_T + \boldsymbol{\Omega}_T\times(\mathbf{r}-\mathbf{r}_0) \tag{1.6}$$

where v_T, $\boldsymbol{\Omega}_T$ and $\mathbf{r}_0$ are constant vectors and $\mathbf{r} = x_1\mathbf{i}_1 + x_2\mathbf{i}_2$ is the radius vector of a point in the middle plane. Using formulae (1.3) and (1.5) as it applies to equality (1.6), we obtain the relations

$$\boldsymbol{\varepsilon} = \mathbf{K} = 0, \quad \boldsymbol{\Omega} = \boldsymbol{\Omega}_T,$$

which explain the geometrical meaning of the rotation vector $\boldsymbol{\Omega}$ and the tensors $\boldsymbol{\varepsilon}$ and $\mathbf{K}$.

We will assume that the plane domain σ, occupied in the plane of the elastic plate, is multiply connected and homeomorphic to a circle with circular apertures. We will denote the outer contour of the domain σ by γ_0 and the contours of the apertures by γ_k ($k = 1, 2, \ldots, N$). We now formulate the problem of determining of the displacement field $v(\mathbf{r})$ in the multiply connected domain σ using the single-valued strain tensor field $\boldsymbol{\varepsilon}$ and the flexural deformation tensor field $\mathbf{K}$, which are prescribed in σ. Here, the function $\boldsymbol{\varepsilon}(\mathbf{r})$ is assumed to be doubly differentiable and the function $\mathbf{K}(\mathbf{r})$ is assumed to be differentiable.

From relations (1.3) and (1.5), we have

$$\nabla\boldsymbol{\upsilon} = \boldsymbol{\varepsilon} - \mathbf{g}\times\boldsymbol{\Omega} \tag{1.7}$$
$$\nabla\boldsymbol{\Omega} = -\mathbf{K}\cdot\mathbf{e} + \boldsymbol{\gamma}\otimes\mathbf{i}_3, \quad \boldsymbol{\gamma}\equiv\nabla\cdot(\mathbf{e}\cdot\boldsymbol{\varepsilon}) \tag{1.8}$$

Integrating equation (1.7), we obtain

$$\boldsymbol{\upsilon} = \boldsymbol{\upsilon}_0 + \int_{\mathbf{r}_0}^{\mathbf{r}}\boldsymbol{\varepsilon}\cdot d\mathbf{r} + \int_{\mathbf{r}_0}^{\mathbf{r}}\boldsymbol{\Omega}\times d\mathbf{r}, \quad \boldsymbol{\upsilon}_0 = \boldsymbol{\upsilon}(\mathbf{r}_0) \tag{1.9}$$

where $\mathbf{r}_0$ is the radius vector of the point chosen as the initial point.

We transform expression (1.9) as follows:

$$\boldsymbol{\upsilon} = \boldsymbol{\upsilon}_0 + \int_{\mathbf{r}_0}^{\mathbf{r}}\boldsymbol{\varepsilon}'\cdot d\mathbf{r}' + \boldsymbol{\Omega}_0\times(\mathbf{r}-\mathbf{r}_0) + \int_{\mathbf{r}_0}^{\mathbf{r}}d\boldsymbol{\Omega}'\times(\mathbf{r}-\mathbf{r}_0)$$
$$\boldsymbol{\Omega}_0 = \boldsymbol{\Omega}(\mathbf{r}_0), \quad \boldsymbol{\varepsilon}' = \boldsymbol{\varepsilon}(\mathbf{r}'), \quad \boldsymbol{\Omega}' = \boldsymbol{\Omega}(\mathbf{r}') \tag{1.10}$$

where the radius vector of the current point on the integration curve is marked with a prime.

Integrating Eq. (1.8), we find

$$\boldsymbol{\Omega}(\mathbf{r}) = \boldsymbol{\Omega}_0 + \int_{\mathbf{r}_0}^{\mathbf{r}}d\mathbf{r}\cdot(-\mathbf{K}\cdot\mathbf{e} + \boldsymbol{\gamma}\otimes\mathbf{i}_3) \tag{1.11}$$

Since $d\boldsymbol{\Omega} = d\mathbf{r}\cdot\nabla\boldsymbol{\Omega}$, from relations (1.8) and (1.10) we obtain

$$\boldsymbol{\upsilon}(\mathbf{r}) = \boldsymbol{\upsilon}_0 + \boldsymbol{\Omega}_0\times(\mathbf{r}-\mathbf{r}_0) + \int_{\mathbf{r}_0}^{\mathbf{r}}d\mathbf{r}'\cdot[\boldsymbol{\varepsilon}' - (\mathbf{K}'\cdot\mathbf{e} - \boldsymbol{\gamma}'\otimes\mathbf{i}_3)\times(\mathbf{r}-\mathbf{r}')] \tag{1.12}$$

Formula (1.12) gives the solution of the problem of determining the field of displacements of the plate using the prescribed deformation and flexural deformation fields. A necessary and sufficient condition for the integrals in equalities (1.11) and (1.12) to be independent of the integration path in a simply connected domain σ is satisfaction of the vector compatibility equation

$$\nabla \cdot \mathbf{e} \cdot (-\mathbf{K} \cdot \mathbf{e} + \boldsymbol{\gamma} \otimes \mathbf{i}_3) = 0 \tag{1.13}$$

This equation splits into two independent equations: the St. Venant compatibility equation in the plane theory of elasticity

$$\nabla \cdot \mathbf{e} \cdot (\nabla \cdot \mathbf{e} \cdot \boldsymbol{\varepsilon}) = 0 \tag{1.14}$$

and the compatibility equation for flexural deformations

$$\nabla \cdot \mathbf{e} \cdot \mathbf{K} = 0 \tag{1.15}$$

which is equivalent to two scalar relations.

If the initial values of the displacements and rotations, υ_0 and $\mathbf{\Omega}_0$, are given, then, when conditions (1.13) are satisfied, expressions (1.11) and (1.12) determine the single-valued functions $\mathbf{\Omega}(\mathbf{r})$ and $\upsilon(\mathbf{r})$ in the simply connected domain σ. In the case of a multiply connected domain, the property of uniqueness of these functions is, generally speaking, lost. The multivalence can be removed by transforming the domain σ into a simply connected domain by making the required number of cuts (partitions). In this case, the values of the rotations and displacements on the different surfaces of a cut will differ. It follows from relations (1.11) and (1.12) that the discontinuity in the functions $\mathbf{\Omega}$ and υ when each cut is intersected is described by the formulae

$$\mathbf{\Omega}_+ - \mathbf{\Omega}_- = \mathbf{Q}_k, \quad \boldsymbol{\upsilon}_+ - \boldsymbol{\upsilon}_- = \boldsymbol{\beta}_k + \mathbf{Q}_k \times \mathbf{r} \tag{1.16}$$

where $\mathbf{Q}_k$ and $\boldsymbol{\beta}_k$ (k = 1, 2, . . ., N) are vectors which are constant for a given cut and are known as the Frank vector and the Burgers vector respectively. These vectors are independent of the choice of the system of cuts and, according to relations (1.11) and (1.12), they are expressed in terms of the deformation and flexural deformation fields by the formulae

$$\mathbf{Q}_k = \mathbf{q}_k + \chi_k \mathbf{i}_3, \ \ \mathbf{q}_k \cdot \mathbf{i}_3 = 0; \quad \mathbf{q}_k = \mathbf{e} \cdot \oint\limits_{\Gamma_k} \mathbf{K} \cdot d\mathbf{r}, \quad \chi_k = \oint\limits_{\Gamma_k} \boldsymbol{\gamma} \cdot d\mathbf{r} \tag{1.17}$$

$$\boldsymbol{\beta}_k = \mathbf{a}_k + b_k \mathbf{i}_3; \quad \mathbf{a}_k = \oint\limits_{\Gamma_k} (\boldsymbol{\varepsilon} + \mathbf{e} \cdot \mathbf{r} \otimes \boldsymbol{\gamma}) \cdot d\mathbf{r}, \quad b_k = -\oint\limits_{\Gamma_k} \mathbf{r} \cdot \mathbf{K} \cdot d\mathbf{r} \tag{1.18}$$

Here Γ_k is any closed contour encompassing just one k-th aperture. By virtue of equalities (1.16), a Volterra dislocation, that is, an isolated defect corresponding to the k-th aperture, is characterized by six parameters. According to the accepted terminology,[6] in the special case when $\mathbf{Q}$ = 0, the defect is called a translational dislocation or, simply, a dislocation, but cases when $\mathbf{Q} \neq 0$ are called a combination of a dislocation and a disclination.

When there are no isolated disclinations ($\mathbf{Q}_k$ = 0), the field of the rotations $\mathbf{\Omega}$ will be single-valued in a multiply connected domain and the formula

$$\boldsymbol{\beta}_k = \oint\limits_{\Gamma_k} (\boldsymbol{\varepsilon} \cdot d\mathbf{r} + \mathbf{\Omega} \times d\mathbf{r})$$

which follows from the equality (1.9), holds for the Burgers vectors together with the expressions (1.18).

When determining the stress-strain state of a multiply connected plate with specified characteristics of the isolated defects $\boldsymbol{\beta}_k$, $\mathbf{Q}_k$ (k = 1, 2, . . ., N), it is necessary to add integral relations (1.17) and (1.18) to equilibrium equations (1.1), constitutive relations (1.2), compatibility equations (1.13) and the boundary conditions on the boundary $\partial\sigma$ of the domain σ.

The problem of the equilibrium of a linearly elastic plate splits into two independent problems: the problem of the plane stress state in which w = 0, $\mathbf{K} = \mathbf{M}$ = 0 and a flexural problem in which $\mathbf{u}$ = 0, $\boldsymbol{\varepsilon} = \mathbf{T}$ = 0. This also refers to the case of a plate with dislocations and disclinations. A wedge disclination ($\mathbf{q}_k$ = 0, $\chi_k \neq 0$) and an edge dislocation (b_k = 0, $\mathbf{a}_k \neq 0$) induce a plane stress state but a twisting disclination (χ_k = 0, $\mathbf{q}_k \neq 0$) and a screw dislocation ($\mathbf{a}_k$ = 0, $b_k \neq 0$) cause the plate to bend.

2. Bending of an annular plate with a screw dislocation and a twisting disclination

We will consider the problem of an isolated defect in a plate in the shape of a circular ring. In this case, the domain σ is prescribed in the polar coordinates r, φ by the inequalities $r_1 \le r \le r_0$, $0 \le \varphi \le 2\pi$. Since the solution of the problem of a boundary dislocation and a wedge dislocation, which give rise to a plane stress state of an annular plate, is known,[7,8] we will consider the case when $\mathbf{a}$ = 0, χ = 0, that is, to the case of the bending of a plate caused by a screw dislocation and a twisting disclination. We will represent the moment tensor and the bending deformation tensor by their expansions in the polar coordinate basis $\mathbf{e}_r$, $\mathbf{e}_\varphi$

$$\begin{aligned}
\mathbf{M} &= M_{rr}\mathbf{e}_r \otimes \mathbf{e}_r + M_{r\varphi}(\mathbf{e}_r \otimes \mathbf{e}_\varphi + \mathbf{e}_\varphi \otimes \mathbf{e}_r) + M_{\varphi\varphi}\mathbf{e}_\varphi \otimes \mathbf{e}_\varphi \\
\mathbf{K} &= K_{rr}\mathbf{e}_r \otimes \mathbf{e}_r + K_{r\varphi}(\mathbf{e}_r \otimes \mathbf{e}_\varphi + \mathbf{e}_\varphi \otimes \mathbf{e}_r) + K_{\varphi\varphi}\mathbf{e}_\varphi \otimes \mathbf{e}_\varphi \\
\mathbf{e}_r &= \mathbf{i}_1 \cos\varphi + \mathbf{i}_2 \sin\varphi, \quad \mathbf{e}_\varphi = -\mathbf{i}_1 \sin\varphi + \mathbf{i}_2 \cos\varphi
\end{aligned} \tag{2.1}$$

Expressing the tensor $\mathbf{M}$ in terms of the tensor $\mathbf{K}$ using Eq. (1.4), the second equilibrium equation of (1.1) can be written in terms of the components of the bending deformation.

Assuming that the annular plate is free from external loads and adding the compatibility equation (1.15) to the equilibrium equation, we obtain a system of equations in polar coordinates with the unknowns K_{rr}, $K_{r\varphi}$ and $K_{\varphi\varphi}$

$$\frac{\partial^2 K_{rr}}{\partial r^2} + \nu\frac{\partial^2 K_{\varphi\varphi}}{\partial r^2} + \frac{2(1-\nu)}{r}\frac{\partial^2 K_{r\varphi}}{\partial r\partial\varphi} + \frac{1}{r^2}\frac{\partial^2 K_{\varphi\varphi}}{\partial\varphi^2} + \frac{\nu}{r^2}\frac{\partial^2 K_{rr}}{\partial\varphi^2}$$
$$+\frac{2-\nu}{r}\frac{\partial K_{rr}}{\partial r} + \frac{2\nu-1}{r}\frac{\partial K_{\varphi\varphi}}{\partial r} + \frac{2(1-\nu)}{r^2}\frac{\partial K_{r\varphi}}{\partial\varphi} = 0 \tag{2.2}$$

$$\frac{\partial K_{r\varphi}}{\partial r} + \frac{2}{r}K_{r\varphi} - \frac{1}{r}\frac{\partial K_{rr}}{\partial\varphi} = 0 \tag{2.3}$$

$$\frac{\partial K_{\varphi\varphi}}{\partial r} + \frac{1}{r}K_{\varphi\varphi} - \frac{1}{r}\frac{\partial K_{r\varphi}}{\partial\varphi} - \frac{1}{r}K_{rr} = 0 \tag{2.4}$$

In the theory of the bending of a plate, the boundary conditions on a boundary which is load-free have the form[4]

$$M_{nn} = 0, \quad \mathbf{n}\cdot(\nabla\cdot\mathbf{M}) + \frac{\partial M_{nt}}{\partial s} = 0$$
$$M_{nn} = \mathbf{n}\cdot\mathbf{M}\cdot\mathbf{n}, \quad M_{nt} = \mathbf{n}\cdot\mathbf{M}\cdot\mathbf{t} \tag{2.5}$$

Here, **n** and **t** are the unit vectors of the normal and the tangent to the boundary contour and s is the current length of the arc on the contour. In the case of an annular plate, boundary conditions (2.5), which are written in terms of the components of the bending deformation tensor, lead to the following constraints when $r = r_1$ and $r = r_0$

$$K_{rr} + \nu K_{\varphi\varphi} = 0 \tag{2.6}$$

$$\frac{\partial}{\partial r}(K_{rr} + \nu K_{\varphi\varphi}) - \frac{1}{r}(K_{\varphi\varphi} + \nu K_{rr}) + \frac{2(1-\nu)}{r}\frac{\partial K_{r\varphi}}{\partial\varphi} = 0 \tag{2.7}$$

We will take as the closed contour in the integral conditions (1.17) and (1.18), a circle of arbitrary radius r, subject to the condition that $r_1 \le r \le r_0$.

Since the tensor **K** is a single valued function in an annular domain, its components are periodic functions of the angular displacement φ and can be represented in the form of complex Fourier series

$$K_{rr} = \sum_{n=-\infty}^{\infty} K_{rr}^{(n)}(r)e^{in\varphi}, \quad K_{r\varphi} = \sum_{n=-\infty}^{\infty} K_{r\varphi}^{(n)}(r)e^{in\varphi}, \quad K_{\varphi\varphi} = \sum_{n=-\infty}^{\infty} K_{\varphi\varphi}^{(n)}(r)e^{in\varphi} \tag{2.8}$$

Substituting expressions (2.8) into Eqs (2.2) - (2.4), we obtain a denumerable set of systems of ordinary differential equations

$$\frac{d^2K_{rr}^{(n)}}{dr^2} + \nu\frac{d^2K_{\varphi\varphi}^{(n)}}{dr^2} + \frac{2(1-\nu)in}{r}\frac{d^2K_{r\varphi}^{(n)}}{dr} - \frac{n^2}{r^2}K_{\varphi\varphi}^{(n)} - \frac{\nu n^2}{r^2}K_{rr}^{(n)}$$
$$+\frac{(2-\nu)}{r}\frac{dK_{rr}^{(n)}}{dr} + \frac{(2\nu-1)}{r}\frac{dK_{\varphi\varphi}^{(n)}}{dr} + \frac{2(1-\nu)in}{r^2}K_{r\varphi}^{(n)} = 0 \tag{2.9}$$

$$\frac{dK_{r\varphi}^{(n)}}{dr} + \frac{2}{r}K_{r\varphi}^{(n)} - \frac{in}{r}K_{rr}^{(n)} = 0 \tag{2.10}$$

$$\frac{dK_{\varphi\varphi}^{(n)}}{dr} + \frac{1}{r}K_{\varphi\varphi}^{(n)} - \frac{in}{r}K_{r\varphi}^{(n)} - \frac{1}{r}K_{rr}^{(n)} = 0 \tag{2.11}$$

From equalities (2.6) - (2.8), we derive the boundary conditions for this system when $r = r_1$ and $r = r_0$

$$K_{rr}^{(n)} + \nu K_{\varphi\varphi}^{(n)} = 0 \tag{2.12}$$

$$\frac{d}{dr}(K_{rr}^{(n)} + \nu K_{\varphi\varphi}^{(n)}) - \frac{1}{r}(\nu K_{rr}^{(n)} + K_{\varphi\varphi}^{(n)}) + \frac{2(1-\nu)in}{r}K_{r\varphi}^{(n)} = 0 \tag{2.13}$$

We now apply the integral relations which express the components of the Burgers and Frank vectors in terms of the bending deformation tensor field. On the basis of equalities (1.18) and (2.8), for a circular contour of radius r we have

$$b = -\int_0^{2\pi} r^2 K_{r\varphi}(r,\varphi)d\varphi = -r^2\int_0^{2\pi}\sum_{n=-\infty}^{\infty} K_{r\varphi}^{(n)}(r)e^{in\varphi}d\varphi = -2\pi r^2 K_{r\varphi}^{(0)}(r)$$

from which we find

$$K_{r\varphi}^{(0)}(r) = -\frac{b}{2\pi r^2} \tag{2.14}$$

According to formulae (1.17), the Frank vector of a twisting disclination has the form

$$\mathbf{q} = q_1\mathbf{i}_1 + q_2\mathbf{i}_2$$
$$q_1 = r\int_0^{2\pi}(K_{r\varphi}\sin\varphi + K_{\varphi\varphi}\cos\varphi)d\varphi, \quad q_2 = r\int_0^{2\pi}(K_{\varphi\varphi}\sin\varphi - K_{r\varphi}\cos\varphi)d\varphi \tag{2.15}$$

Using series (2.8), we obtain from this that

$$K_{\varphi\varphi}^{(1)} + iK_{r\varphi}^{(1)} = \frac{q_1 - iq_2}{2\pi r}, \quad K_{\varphi\varphi}^{(-1)} - iK_{r\varphi}^{(-1)} = \frac{q_1 + iq_2}{2\pi r} \tag{2.16}$$

It follows from equalities (2.14) and (2.16) that, when $n = \pm 2, \pm 3, \ldots$, the systems of equations (2.9) - (2.11) with the boundary conditions (2.12) and (2.13) only have a trivial solution. Hence, among the coefficients of the Fourier series (2.8), only the coefficients with the numbers $n = -1$, $n = 0$, $n = 1$ are non-zero. These coefficients are uniquely defined as solutions of the system of equations (2.9) - (2.11) which satisfy relations (2.14) and (2.16) and boundary conditions (2.12) and (2.13). Finally, we find the solution of the problem on the bending of an annular plate with a screw dislocation and a twisting disclination

$$K_{rr} = \left(3Hr - \frac{F}{r} + \frac{G}{r^3}\right)(\mathbf{q}\cdot\mathbf{e}_r)$$
$$K_{\varphi\varphi} = \left(Hr + \frac{1-F}{r} - \frac{G}{r^3}\right)(\mathbf{q}\cdot\mathbf{e}_r)$$
$$K_{r\varphi} = -\frac{b}{2\pi r^2} + \left(Hr - \frac{F}{r} - \frac{G}{r^3}\right)(\mathbf{q}\cdot\mathbf{e}_\varphi)$$
$$H = \frac{(1-\nu)^2}{4\pi(3+\nu)(r_0^2 + r_1^2)}, \quad F = \frac{1+\nu}{4\pi}, \quad G = \frac{(1-\nu)r_0^2 r_1^2}{4\pi(r_0^2 + r_1^2)} \tag{2.17}$$

Using expressions (1.4) and (2.17), we calculate the components of the tensor of the moments which are caused by the isolated defect. We obtain

$$M_{rr} = D_1\left[-\frac{G_1}{r^3} + \frac{1}{r} - H_1 r\right](\mathbf{q}\cdot\mathbf{e}_r)$$
$$M_{\varphi\varphi} = D_1\left[\frac{G_1}{r^3} - \frac{3+\nu}{(1-\nu)r} - \frac{1+3\nu}{3+\nu}H_1 r\right](\mathbf{q}\cdot\mathbf{e}_r)$$
$$M_{r\varphi} = \frac{D(1-\nu)b}{2\pi r^2} + D_1\left[\frac{G_1}{r^3} + \frac{1+\nu}{(1-\nu)r} - \frac{1-\nu}{3+\nu}H_1 r\right](\mathbf{q}\cdot\mathbf{e}_\varphi)$$
$$D_1 = \frac{D(1-\nu)^2}{4\pi}, \quad G_1 = \frac{r_0^2 r_1^2}{r_0^2 + r_1^2}, \quad H_1 = \frac{1}{r_0^2 + r_1^2} \tag{2.18}$$

Using formulae (1.11), (1.12) and (2.17), we find the multivalued rotation field and bending function of the annular plate

$$\mathbf{\Omega}(r,\varphi) = \theta_\varphi\mathbf{e}_r - \theta_e\mathbf{e}_\varphi$$
$$\theta_r = \left(\frac{3H}{2}r^2 - F\ln r - \frac{G}{2r^2}\right)(\mathbf{q}\cdot\mathbf{e}_r) - \frac{\varphi}{2\pi}(\mathbf{q}\cdot\mathbf{e}_\varphi) + \mathbf{c}\cdot\mathbf{e}_r \tag{2.19}$$
$$\theta_\varphi = \frac{b}{2\pi r} + \left(\frac{H}{2}r^2 + F(1-\ln r) + \frac{G}{2r^2} - \frac{1}{2\pi}\right)(\mathbf{q}\cdot\mathbf{e}_\varphi) + \frac{\varphi}{2\pi}(\mathbf{q}\cdot\mathbf{e}_r) + \mathbf{c}\cdot\mathbf{e}_\varphi$$

$$w(r,\varphi) = \left(\frac{H}{2}r^3 + F(r - r\ln r) + \frac{G}{2r}\right)(\mathbf{q}\cdot\mathbf{e}_r) + \frac{\varphi}{2\pi}(b + \mathbf{q}\cdot\mathbf{e}\cdot\mathbf{r}) + \mathbf{c}\cdot\mathbf{r} + d \tag{2.20}$$

Here, $\mathbf{c}$ is a constant plane vector and d is a scalar constant. Since these constants can be arbitrary, the bending of the plate is determined, apart from its rigid motion.

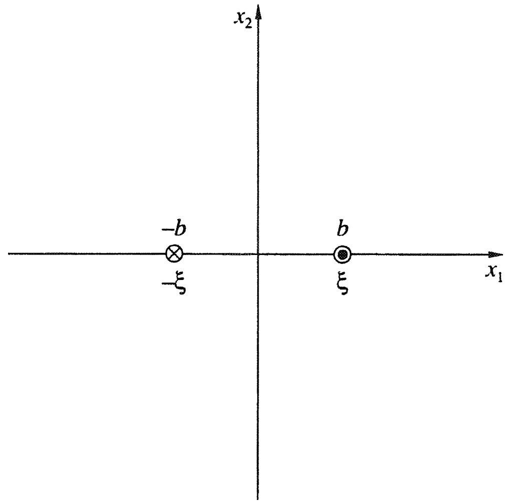

Fig. 1.

3. A concentrated defect in an unbounded plate

If, in the solution obtained in Section 2, the inner radius of the annular plate r_1 tends to zero and the outer radius r_0 tends to infinity, we obtain the solution of the problem of a concentrated defect in an unbounded plate situated at the origin of coordinates. The defect is a combination of a screw dislocation and a twisting disclination. The line (axis) of this defect can be represented as a straight line which is orthogonal to the plane of the plate. On taking the above-mentioned limit in formulae (2.17), we obtain

$$
\begin{aligned}
K_{rr} &= -\frac{1+\nu}{4\pi r}(q_1\cos\varphi + q_2\sin\varphi)\\
K_{\varphi\varphi} &= \frac{3-\nu}{\pi r}(q_1\cos\varphi + q_2\sin\varphi)\\
K_{r\varphi} &= -\frac{b}{2\pi r^2} - \frac{1+\nu}{\pi r}(q_2\cos\varphi - q_1\sin\varphi)\\
\mathbf{q} &= q_1\mathbf{i}_1 + q_2\mathbf{i}_2
\end{aligned}
\tag{3.1}
$$

Here q_1, q_2 are the components of the Frank vector in the constant basis of Cartesian coordinates.

Converting to Cartesian coordinates in equalities (3.1), we find

$$
\begin{aligned}
K_{11} &= \frac{4bx_1x_2 + q_1(-a_1x_1^3 + a_2x_1x_2^2) + q_2(a_1x_1^2x_2 + a_3x_2^3)}{4\pi(x_1^2+x_2^2)^2}\\
K_{22} &= \frac{-4bx_1x_2 + q_1(a_3x_1^3 + a_1x_1x_2^2) + q_2(a_2x_1^2x_2 - a_1x_2^3)}{4\pi(x_1^2+x_2^2)^2}\\
K_{12} &= -\frac{2b(x_1^2 - x_2^2) + q_1(a_1x_2^3 + a_3x_1^2x_2) + q_2(a_1x_1^3 + a_3x_1x_2^2)}{4\pi(x_1^2+x_2^2)^2}\\
a_1 &= 1+\nu, \quad a_2 = 1-3\nu, \quad a_3 = 3-\nu, \quad K_{\alpha\beta} = \mathbf{i}_\alpha\cdot\mathbf{K}\cdot\mathbf{i}_\beta, \quad \alpha,\beta = 1,2
\end{aligned}
\tag{3.2}
$$

We will write the components of the moment tensor in the case of an isotropic material, which are defined by expressions (1.4) and (3.2), separately for the cases of a screw dislocation and a twisting disclination with a Frank vector directed along the x_1 axis (Fig. 1)

$$
M_{11} = -M_{22} = -D(1-\nu)b\frac{x_1x_2}{\pi(x_1^2+x_2^2)^2}, \quad M_{12} = D(1-\nu)b\frac{x_1^2 - x_2^2}{2\pi(x_1^2+x_2^2)^2}
\tag{3.3}
$$

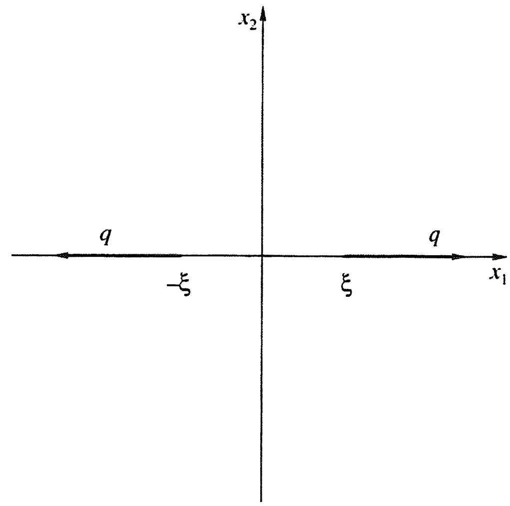

Fig. 2.

$$M_{11} = D(1-\nu)^2 q \frac{x_1(x_1^2 - x_2^2)}{4\pi(x_1^2 + x_2^2)^2}$$

$$M_{22} = -D(1-\nu) q \frac{x_1[(3+\nu)x_1^2 + (1+3\nu)x_2^2]}{4\pi(x_1^2 + x_2^2)^2}$$

$$M_{12} = D(1-\nu) q \frac{x_2[(3-\nu)x_1^2 + (1+\nu)x_2^2]}{4\pi(x_1^2 + x_2^2)^2} \tag{3.4}$$

It can be seen from expressions (3.3) and (3.4) that the order of the singularity in the moments on the axis of the screw dislocation, that is, when $r \to 0$, is higher than the order of the singularity on the axis of the twisting disclination. The singularity generated by a screw dislocation in the thin plate is non-integrable over the area, unlike the stress distribution close to the axis of a linear defect in a linearly elastic spatial medium in which, as is well known,[6] the stress fields caused both by a screw dislocation and a twisting disclination have an r^{-1} order singularity as the axis of the defect is approached.

In order to obtain the components of the tensor fields of the bending deformation and the moments caused by a concentrated defect located at the point x_1^0, x_2^0 of an infinite plate, it is sufficient to replace x_α by $(x_\alpha - x_\alpha^0)$, $\alpha = 1, 2$ in formulae (3.2) - (3.4).

4. The dipoles of dislocations and disclinations

Suppose there are two concentrated defects at the points $(\xi, 0)$ and $(-\xi, 0)$ in an unbounded plate with dislocation and disclination parameters which are equal in modulus both opposite in sign. According to the principle of superposition, the effect from a pair of defects will be equal to the sum of the effects from each of them. We shall consider three cases.

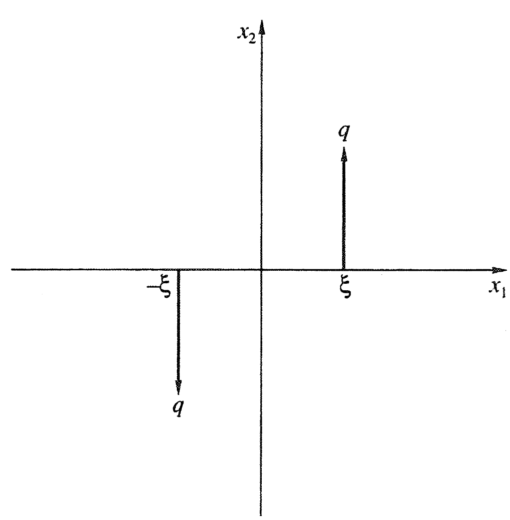

Fig. 3.

4.1. The screw dislocation dipole (Fig. 1)

Consider the limiting state of a plate when $\xi \to 0$, $b \to \infty$, where $2\xi b = m_0 = \text{const}$. After reduction using relations (3.2) and (3.3) we obtain

$$K_{11} = -K_{22} = m_0 \frac{x_2(3x_1^2 - x_2^2)}{\pi(x_1^2 + x_2^2)^3}, \quad K_{12} = m_0 \frac{x_1(3x_2^2 - x_1^2)}{\pi(x_1^2 + x_2^2)^3}$$

$$M_{11} = -M_{22} = -D(1-\nu)K_{11}, \quad M_{12} = D(1-\nu)K_{12} \tag{4.1}$$

4.2. The dipole of a twisting disclination of the first type

Two twisting disclinations with Frank vectors lying on a line passing through the disclinations (Fig. 2). Using formulae (3.2) and (3.4) and taking the limit $\xi \to 0$, $q \to \infty$ with the condition that $2\xi q = m_1 = \text{const}$, we obtain

$$K_{11} = -m_1 \frac{(1+\nu)x_1^4 - 6(1-\nu)x_1^2x_2^2 + (1-3\nu)x_2^4}{4\pi(x_1^2 + x_2^2)^3}$$

$$K_{22} = m_1 \frac{(3-\nu)x_1^4 - 6(1-\nu)x_1^2x_2^2 - (1+\nu)x_2^4}{4\pi(x_1^2 + x_2^2)^3}$$

$$K_{12} = m_1 \frac{x_1x_2(-(3-\nu)x_1^2 + (1-3\nu)x_2^2)}{2\pi(x_1^2 + x_2^2)^3}$$

$$M_{11} = D(1-\nu)^2 m_1 \frac{x_1^4 - 6x_1^2x_2^2 + x_2^4}{4\pi(x_1^2 + x_2^2)^3}$$

$$M_{22} = D(1-\nu)m_1 \frac{-(3+\nu)x_1^4 + 6(1-\nu)x_1^2x_2^2 + (1+3\nu)x_2^4}{4\pi(x_1^2 + x_2^2)^3}$$

$$M_{12} = -D(1-\nu)K_{12} \tag{4.2}$$

4.3. A disclination dipole of the second type

Two twisting disclinations with Frank vectors which are orthogonal to the segment connecting the disclinations (Fig. 3). Using formula (3.2) and (3.4), taking the limit when $\xi \to 0$, $q \to \infty$, for which $2\xi q = m_2 = \text{const}$, we obtain the bending deformation and moment fields

$$K_{11} = m_2 \frac{x_1x_2((1+\nu)x_1^2 + (5-3\nu)x_2^2)}{2\pi(x_1^2 + x_2^2)^3}$$

$$K_{22} = m_2 \frac{x_1x_2((1-3\nu)x_1^2 - (3-\nu)x_2^2)}{2\pi(x_1^2 + x_2^2)^3}$$

$$K_{12} = -m_2 \frac{(1+\nu)x_1^4 + 6(1-\nu)x_1^2x_2^2 - (3-\nu)x_2^4}{4\pi(x_1^2 + x_2^2)^3}$$

$$M_{11} = -D(1-\nu)m_2 \frac{x_1x_2[(1+3\nu)x_1^2 + (5-\nu)x_2^2]}{2\pi(x_1^2 + x_2^2)^3}$$

$$M_{22} = D(1-\nu)^2 m_2 \frac{x_1x_2(3x_2^2 - x_1^2)}{2\pi(x_1^2 + x_2^2)^3}$$

$$M_{12} = -D(1-\nu)K_{12} \tag{4.3}$$

We shall call the quantity m_2 a dipole moment.

Using the first three formulae of (4.3), we determine the deformed state of an infinite plate caused by the superposition of two mutually perpendicular disclination dipoles of the second type with the same moment m (Fig. 4). Comparing the results with relation (3.3), we see

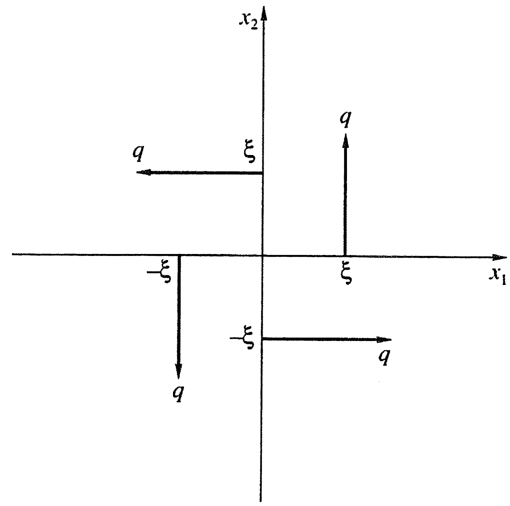

Fig. 4.

that the combination of orthogonal twisting disclination dipoles is equivalent to a screw dislocation and that the magnitude of the Burgers dislocation vector is equal to the combined dipole moment, that is, $b = 2m$.

5. A Continuous distribution of dislocations and disclinations in an elastic plate

If the number of dislocations and disclinations in a bounded part of a plate is extremely large, it is more convenient to change to a continuous distribution of the defects. The resolvents describing the equilibrium of an elastic plate with continuously distributed dislocations and disclinations can be derived by taking the limit as a discrete set of Volterra dislocations changes to their continuous distribution.

We will first consider the following auxiliary problem: it is acquired to determine the displacement field $\boldsymbol{v}(x_1, x_2)$ in a multiply connected plane domain using the specified single-valued and differentiable fields of the strain tensor $\boldsymbol{\varepsilon}$ and the rotation vector $\boldsymbol{\Omega}$. From equality (1.7), we have

$$\boldsymbol{\upsilon}(\mathbf{r}) = \int_{\mathbf{r}_0}^{\mathbf{r}} d\mathbf{r} \cdot (\boldsymbol{\varepsilon} - \mathbf{g} \times \boldsymbol{\Omega}) + \boldsymbol{\upsilon}(\mathbf{r}_0) \tag{5.1}$$

The necessary and sufficient condition for the curvilinear integral in relation (5.1) to be independent of the choice of the path in a simply connected domain is the equality

$$\nabla \cdot \mathbf{e} \cdot (\boldsymbol{\varepsilon} - \mathbf{g} \times \boldsymbol{\Omega}) = 0 \tag{5.2}$$

If the domain is multiply connected, then expression (5.1) gives, generally speaking, a multivalued function. After converting the domain into a simply connected domain by making cuts, the values of the function $\boldsymbol{v}_{\pm}$ on the opposite faces of each cut can only differ by a constant quantity: $\boldsymbol{v}_{+} - \boldsymbol{v}_{-} = \boldsymbol{\beta}_k$. By virtue of equality (5.1), the Burgers vectors are expressed in terms of the fields $\boldsymbol{\varepsilon}$ and $\boldsymbol{\Omega}$ using the formula

$$\boldsymbol{\beta}_k = \oint_{\Gamma_k} d\mathbf{r} \cdot (\boldsymbol{\varepsilon} - \mathbf{g} \times \boldsymbol{\Omega}) \tag{5.3}$$

According to expression (5.3), the total Burgers vector of the discrete set of N isolated translational dislocations in an arbitrary domain σ_* is given by the formula

$$\mathbf{B} = \sum_{k=1}^{N} \boldsymbol{\beta}_k = \sum_{k=1}^{N} \oint_{\Gamma_k} d\mathbf{r} \cdot (\boldsymbol{\varepsilon} - \mathbf{g} \times \boldsymbol{\Omega}) \tag{5.4}$$

By virtue of the known properties of curvilinear integrals, the sum of the integrals along the contours Γ_k can be replaced by a single integral along a closed contour Γ which encompasses all of the contours Γ_k, that is, all the dislocations in the domain σ_*:

$$\mathbf{B} = \oint_{\Gamma} d\mathbf{r} \cdot (\boldsymbol{\varepsilon} - \mathbf{g} \times \boldsymbol{\Omega}) \tag{5.5}$$

In order to take the limit of a simply connected domain with continuously distributed dislocations, we direct the diameters of the apertures to zero and convert the contour integral in equality (5.5) into an integral over the area of the resulting simply connected domain σ_*. For the total Burgers vector, we obtain the expression

$$\mathbf{B} = \iint_{\sigma_*} \boldsymbol{\alpha} \, d\sigma \tag{5.6}$$

where

$$\boldsymbol{\alpha} = \boldsymbol{\gamma} - \nabla \cdot (\mathbf{e} \times \boldsymbol{\Omega}) = \boldsymbol{\gamma} - (\nabla \cdot \boldsymbol{\Omega})\mathbf{i}_3 - (\nabla \boldsymbol{\Omega}) \cdot \mathbf{i}_3; \quad \boldsymbol{\gamma} \equiv \nabla \cdot \mathbf{e} \cdot \boldsymbol{\varepsilon} \tag{5.7}$$

We will call the vector integrand (5.7) the dislocation density. This definition is motivated by the fact that the total Burgers vector of a set of dislocations in any domain σ_* is equal to the integral of the dislocation density over the area of this domain. When $\boldsymbol{\alpha} = 0$, relation (5.7) becomes Eq. (5.2).

We will now assume that a plate, containing continuously distributed dislocations, occupies a multiply connected domain and formulate the problem of determining the rotation field $\boldsymbol{\Omega}$ in a multiply connected domain using the specified, single-valued and differentiable fields $\boldsymbol{\gamma}$, $\mathbf{K}$ and $\boldsymbol{\alpha}$. We now drop the requirement that the rotations are single-valued.

On the basis of relations (1.3), (1.5) and (5.7), we compile the equation for determining the rotation field $\boldsymbol{\Omega}$. The equality

$$\mathbf{g} \cdot \boldsymbol{\alpha} = \boldsymbol{\gamma} - (\nabla \boldsymbol{\Omega}) \cdot \mathbf{i}_3,$$

is a corollary of relation (5.7) and, from this, we find

$$(\nabla \boldsymbol{\Omega}) \cdot \mathbf{i}_3 \otimes \mathbf{i}_3 = \boldsymbol{\gamma} \otimes \mathbf{i}_3 - \mathbf{g} \cdot \boldsymbol{\alpha} \otimes \mathbf{i}_3 \tag{5.8}$$

From the definition of the bending strain tensor (1.3), (1.5) we have

$$(\nabla \boldsymbol{\Omega}) \cdot \mathbf{g} = -\mathbf{K} \cdot \mathbf{e} \tag{5.9}$$

Combining relations (5.8) and (5.9) and taking account of the equality $\mathbf{g} + \mathbf{i}_3 \otimes \mathbf{i}_3 = \mathbf{E}$, where $\mathbf{E}$ is a three-dimensional unit tensor, we obtain the equation for determining the vector field $\boldsymbol{\Omega}$

$$\nabla \boldsymbol{\Omega} = \boldsymbol{\gamma} \otimes \mathbf{i}_3 - \mathbf{K} \cdot \mathbf{e} - (\mathbf{g} \cdot \boldsymbol{\alpha}) \otimes \mathbf{i}_3 \tag{5.10}$$

It is important to note that the density of screw dislocations does not appear on the right-hand side of Eq. (5.10).

Integration of Eq. (5.10) gives

$$\boldsymbol{\Omega}(\mathbf{r}) = \int_{\mathbf{r}_0}^{\mathbf{r}} d\mathbf{r} \cdot \mathbf{L} + \boldsymbol{\Omega}(\mathbf{r}_0); \quad \mathbf{L} = (\boldsymbol{\gamma} - \mathbf{g} \cdot \boldsymbol{\alpha}) \otimes \mathbf{i}_3 - \mathbf{K} \cdot \mathbf{e} \tag{5.11}$$

By virtue of this equality, after the domain has been transformed into a simply connected domain by making cuts, the function $\boldsymbol{\Omega}$ can undergo a constant jump when each of the cuts is intersected:

$$\boldsymbol{\Omega}_+ - \boldsymbol{\Omega}_- = \mathbf{Q}_k, \quad \mathbf{Q}_k = \oint_{\Gamma_k} d\mathbf{r} \cdot \mathbf{L} \tag{5.12}$$

Non-zero constants $\mathbf{Q}_k$ denote the existence of isolated disclinations in a multiply connected plate with continuously distributed dislocations. According to equalities (5.12), the total Frank vector of a discrete set of N disclinations is expressed by the formula

$$\mathbf{Q} = \sum_{k=1}^{N} \mathbf{Q}_k = \sum_{k=1}^{N} \oint_{\Gamma_k} d\mathbf{r} \cdot \mathbf{L} = \oint_{\Gamma} d\mathbf{r} \cdot \mathbf{L} \tag{5.13}$$

Reasoning in a similar manner as before, we change from a discrete set of disclinations to their continuous distribution with a vector density $\boldsymbol{\Lambda}$, for which the following expression is obtained

$$\boldsymbol{\Lambda} = \nabla \cdot \mathbf{e} \cdot (\boldsymbol{\gamma} - \boldsymbol{\alpha}) \otimes \mathbf{i}_3 - \nabla \cdot (\mathbf{e} \cdot \mathbf{K} \cdot \mathbf{e}) \tag{5.14}$$

Representing the vector $\boldsymbol{\Lambda}$ in the form of an expansion

$$\boldsymbol{\Lambda} = \boldsymbol{\lambda} + \tau \mathbf{i}_3, \quad \boldsymbol{\lambda} \cdot \mathbf{i}_3 = 0$$

where $\boldsymbol{\lambda}$ is the twisting disclination density and τ is the wedge disclination density, we split Eq. (5.14) into two independent equations: an incompatibility equation of the plane theory of elasticity

$$\nabla \cdot \mathbf{e} \cdot (\nabla \cdot \mathbf{e} \cdot \boldsymbol{\varepsilon}) = \tau + \nabla \cdot \mathbf{e} \cdot \boldsymbol{\alpha} \tag{5.15}$$

and an equation for the incompatibility of the bending deformations.

$$\nabla \cdot (\mathbf{e} \cdot \mathbf{K} \cdot \mathbf{e}) = -\boldsymbol{\lambda} \tag{5.16}$$

Here, the dislocation density $\boldsymbol{\alpha}$ and the disclination densities $\boldsymbol{\lambda}$ and τ are assumed to be specified functions of the coordinates x_1, x_2.

When there are distributed defects, the compatibility equations (1.14) and (1.15) are not satisfied and the displacement field of the plate $\boldsymbol{v}$ therefore does not exist. This does not interfere with the determination of the stress-strain state of the plate since the tensors $\boldsymbol{\varepsilon}$, $\mathbf{K}$, $\mathbf{T}$, and $\mathbf{M}$ exist as single-valued functions of the coordinates. In order to find them, equilibrium equations (1.1), constitutive relations (1.2) and the boundary conditions on the boundary $\partial\sigma$ have to be added to incompatibility equations (5.15) and (5.16).

Note that, since the screw dislocation density does not appear in the incompatibility equations (5.15) and (5.16), the two-dimensional model of a Kirchhoff plate does not enable us to consider a continuous distribution of screw dislocations. This is also confirmed by the fact that, according to relations (3.3), the solution of the problem of a concentrated screw dislocation has a non-integrable singularity. Continuously distributed screw dislocations can be taken into account in the two-dimensional theory of plates of the Cosserat type.[9]

If the disclination density is equal to zero ($\tau = 0$, $\boldsymbol{\lambda} = 0$) with the condition that $\boldsymbol{\alpha} \neq 0$, then the conditions for Eq. (5.10) to be integrable are satisfied, that is, there is a continuous field of rotations $\boldsymbol{\Omega}$ in the plate but a displacement field $\boldsymbol{v}$ does not exist. In this important special case of distributed defects, that is, of dislocations for which there are no disclinations, the bending deformation compatibility equations (1.15) are satisfied and the metric deformation incompatibility equation (5.15) only contains the dislocation density $\boldsymbol{\alpha}$.

6. The bending of plates with distributed disclinations

In this section, we consider some problems of the bending of plates which contain continuously distributed disclinations and are free from external loads. According to (1.1), (1.4) and (5.16), the system of equations for this problem has the form

$$\nabla \cdot (\nabla \cdot \mathbf{M}) = 0 \tag{6.1}$$

$$\nabla \cdot (\mathbf{e} \cdot \mathbf{K}) = -\mathbf{e} \cdot \boldsymbol{\lambda} \tag{6.2}$$

$$\mathbf{K} = -\frac{1}{D(1-\nu)}\left(\mathbf{M} - \frac{\nu}{1+\nu}\mathbf{g}\,\mathrm{tr}\mathbf{M}\right) \tag{6.3}$$

The equation for the equilibrium of the moments (6.1) can be identically satisfied using the substitution

$$\mathbf{M} = \frac{1}{2}\mathbf{e} \cdot (\nabla\boldsymbol{\psi} + (\nabla\boldsymbol{\psi})^T) \cdot \mathbf{e} \tag{6.4}$$

where $\boldsymbol{\psi}$ is an arbitrary, triply differentiable plane vector field. By analogy with the stress functions in the theory of elasticity, we shall call $\boldsymbol{\psi}$ the moment function. From relations (6.2) and (6.3) we obtain the equation for finding the moment function

$$(1+\nu)\Delta\boldsymbol{\psi} + (1-\nu)\nabla\nabla \cdot \boldsymbol{\psi} = 2D(1-\nu^2)\boldsymbol{\lambda} \tag{6.5}$$

The boundary conditions (2.5) on the boundary of the plate $\partial\sigma$, which is load-free, are written in terms of the function $\boldsymbol{\psi}$ as follows:

$$\frac{\partial\boldsymbol{\psi}}{\partial s} \cdot \mathbf{n} = A, \quad \frac{\partial\boldsymbol{\psi}}{\partial s} \cdot \mathbf{t} = 0 \tag{6.6}$$

where A is a scalar constant. Integrating relation (6.6) along the curve $\partial\sigma$, we obtain

$$\boldsymbol{\psi}(s) = A\mathbf{e} \cdot \mathbf{r} + \mathbf{C} \tag{6.7}$$

where $\mathbf{C}$ is a constant two-dimensional vector. It can be seen that the addition of expression (6.7) to the function $\boldsymbol{\psi}$ has no effect on the value of the tensor $\mathbf{M}$, that is, on the stress state of the plate. Hence, it is possible to put $A = 0$, $\mathbf{C} = 0$ in the case of a simply connected plate, which leads to the boundary condition

$$\boldsymbol{\psi} = 0 \quad \partial\sigma \tag{6.8}$$

We will now consider some problems of the bending of a plate with continuously distributed twisting disclinations.

6.1. A circular plate with axisymmetrically distributed disclinations

Suppose r_0 is the radius of the plate and the components of the disclination density vector in a polar system of coordinates are independent of the angular coordinate:

$$\boldsymbol{\lambda}(r,\varphi) = \lambda_1(r)\mathbf{e}_r + \lambda_2(r)\mathbf{e}_\varphi$$

It is obviously necessary to seek the moment function in the same form

$$\boldsymbol{\psi}(r,\varphi) = \psi_1(r)\mathbf{e}_r + \psi_2(r)\mathbf{e}_\varphi$$

In this case, Eq. (6.5) splits into the two independent scalar equations

$$\psi''_\alpha(r) + \frac{\psi'_\alpha(r)}{r} - \frac{\psi_\alpha(r)}{r^2} = \delta_\alpha\lambda_\alpha(r), \quad \alpha = 1, 2$$

$$\delta_1 = D(1-\nu^2), \quad \delta_2 = 2D(1-\nu)$$

and, solving these, we find

$$\psi_\alpha(r) = c_{\alpha1}r + \frac{c_{\alpha2}}{r} + \frac{\delta_\alpha}{2}\left(r\int_0^r\lambda_\alpha(\rho)d\rho - \frac{1}{r}\int_0^r\lambda_\alpha(\rho)\rho^2 d\rho\right), \quad \alpha = 1, 2 \tag{6.9}$$

In order to avoid the singularity in the components of the moment tensor at the point $r = 0$, as can be seen from the equality (6.4), it is necessary to put

$$c_{12} = c_{22} = 0$$

in solution (6.9).

Using condition (6.8), the boundary conditions, from which the constants c_{11} and c_{21} are found, are written as:

$$\psi_\alpha(r_0) = 0, \quad \alpha = 1, 2 \tag{6.10}$$

We will now consider a special case when the components of the disclination density vector in a polar system of coordinates are constant quantities, that is, $\lambda_\alpha(r) = \lambda_\alpha = \text{const}$. Then,

$$\psi_\alpha(r) = \frac{\delta_\alpha \lambda_\alpha}{3}(r^2 - r_0 r) \tag{6.11}$$

From this and from equality (6.4) we find the components of the moment tensor

$$M_{rr} = \frac{\delta_1\lambda_1}{3}(r_0 - r), \quad M_{\varphi\varphi} = \frac{\delta_1\lambda_1}{3}(r_0 - 2r), \quad M_{r\varphi} = \frac{\delta_2\lambda_2}{6}r \tag{6.12}$$

6.2. An elliptic plate with uniformly distributed disclinations

Suppose the equation of the boundary contour of a plate has the form

$$x_1^2/a^2 + x_2^2/b^2 = 1 \tag{6.13}$$

and the twisting disclination density vector is constant:

$$\boldsymbol{\lambda}(x_1, x_2) = \lambda_1 \mathbf{i}_1 + \lambda_2 \mathbf{i}_2 \tag{6.14}$$

We shall seek the moment function in the following form

$$\boldsymbol{\psi}(x_1, x_2) = \mathbf{A}(x_1^2/a^2 + x_2^2/b^2 - 1) \tag{6.15}$$

Here, $\mathbf{A} = \mathbf{A}_1\mathbf{i}_1 + \mathbf{A}_2\mathbf{i}_2$ is an unknown constant vector, and the boundary conditions (6.8) are automatically satisfied. Substituting expressions (6.14) and (6.15) into Eq. (6.5), we find

$$A_1 = \frac{D(1-\nu^2)\lambda_1 a^2 b^2}{(1+\nu)a^2 + 2b^2}, \quad A_2 = \frac{D(1-\nu^2)\lambda_2 a^2 b^2}{(1+\nu)b^2 + 2a^2} \tag{6.16}$$

Taking account of relations (6.15) and (6.16), from equality (6.4) we obtain the components of the moment tensor

$$M_{11} = -\frac{2A_2}{b^2}x_2, \quad M_{22} = -\frac{2A_1}{a^2}x_1, \quad M_{12} = \frac{A_2}{a^2}x_1 + \frac{A_1}{b^2}x_2 \tag{6.17}$$

In the case of a circular plate ($a = b$), we have

$$M_{11} = -2A_0\lambda_2 x_2, \quad M_{22} = -2A_0\lambda_1 x_1, \quad M_{12} = A_0(\lambda_2 x_1 + \lambda_1 x_2)$$

$$M_{rr} = 0, \quad M_{\varphi\varphi} = -2A_0 r(\boldsymbol{\lambda}\cdot\mathbf{e}_r), \quad M_{r\varphi} = A_0 r(\boldsymbol{\lambda}\cdot\mathbf{e}_\varphi); \quad A_0 = \frac{D(1-\nu^2)}{3+\nu} \tag{6.18}$$

The system of equations (6.1) - (6.3) can also be applied to the problem of a concentrated twisting disclination in an infinite plate. In this case, the disclination density functions λ will be a vector delta-function of two variables, concentrated at the point $x_1 = 0$, $x_2 = 0$. Using a two-dimensional Fourier transformation, it is possible to find the solution of this problem. Omitting the details, we note that this solution is identical to the solution obtained in Section 3 by another method. The correctness of the mathematical model of an elastic plate with distributed disclinations constructed in Section 5 is confirmed by this.

7. A static-geometrical analogy in the theory of plates with dislocations and disclinations

The well-known analogy[10] between the plane problem in the theory of elasticity and the problem of the bending of a plate can be extended to the case of a plate containing dislocations and disclinations. We will consider the problem of the bending of a plate which is free from external loads. when there are continuously distributed disclinations. The equations of this problem consist of relations (6.4), incompatibility equations (5.16) and constitutive relations $\mathbf{M} = \mathbf{M}(\mathbf{K})$. The moment vector function $\boldsymbol{\psi}$ vanishes on the edge of the plate, that is, on the boundary of the domain σ.

Together with the above problem, we will consider, for plates occupying the same domain σ, the problem of the plane stress state when there are no dislocations and disclinations. The deformation of the plate is caused by distributed mass forces acting in its surface. The plate is clamped on the boundary $\partial\sigma$, that is, the displacement vector $\mathbf{u}$ is equal to zero. The system of equations of this problem consists of the first equation of (1.1), the first relation of (1.3) and the constitutive relations, which we represent in the form of the dependence $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}(\mathbf{T})$. We introduce the notation $\mathbf{K}^* = \mathbf{e}\cdot\mathbf{K}\cdot\mathbf{e}$, $\boldsymbol{\varepsilon}^* = \mathbf{e}\cdot\boldsymbol{\varepsilon}\cdot\mathbf{e}$ and write the formulation of the above two boundary-value problems in the form of Table

1, which consists of two columns.

$\mathbf{M} = \frac{1}{2}\mathbf{e}\cdot[\nabla\boldsymbol{\psi} + (\nabla\boldsymbol{\psi})^T]\cdot\mathbf{e}$	$\boldsymbol{\varepsilon}^* = \frac{1}{2}\mathbf{e}\cdot[\nabla\mathbf{u} + (\nabla\mathbf{u})^T]\cdot\mathbf{e}$
$\nabla\cdot\mathbf{K}^* + \boldsymbol{\lambda} = 0$	$\nabla\cdot\mathbf{T} + \mathbf{f} = 0$
$\mathbf{M} = \Phi(\mathbf{K}^*)$	$\boldsymbol{\varepsilon}^* = \Phi_1(\mathbf{T})$
$\boldsymbol{\psi}\vert_{\partial\sigma} = 0$	$\mathbf{u}\vert_{\partial\sigma} = 0$

It is assumed that the problems of the equilibrium of the plate are formulated in dimensionless form, that is, all the quantities in Table 1 are dimensionless. The disclination density $\boldsymbol{\lambda}$ and the distributed load $\mathbf{f}$ are assumed to be specified functions of the coordinates x_1, x_2. We will say that two elastic plates of the same thickness and same shape are conjugate if the function $\boldsymbol{\Phi}$ in the constitutive relations of the first plate is identical to the function $\boldsymbol{\Phi}_1$ in the constitutive relations of the second plate.

The left column of Table 1 becomes into the right column and vice versa with the following reciprocal exchanges

$$\boldsymbol{\psi} \rightleftarrows \mathbf{u},\quad \mathbf{M} \rightleftarrows \boldsymbol{\varepsilon}^*,\quad \mathbf{K}^* \rightleftarrows \mathbf{T},\quad \boldsymbol{\lambda} \rightleftarrows \mathbf{f},\quad \Phi \rightleftarrows \Phi_1$$

It follows from this that, in the case of the condition $\boldsymbol{\lambda} = \mathbf{f}$, the problem of the bending of a plate with distributed disclinations and with a free edge is mathematically equivalent to the problem of the plane stress state of the conjugate plate with a clamped boundary and loaded with mass forces.

As an example, we will find the constitutive relations of mutually conjugate plates in the case of an isotropic material. From relations (1.4), we have

$$\mathbf{M}_0 = -(1-\nu)\mathbf{e}\cdot\mathbf{K}^*\cdot\mathbf{e} + \nu\mathbf{g}\mathrm{tr}\mathbf{K}^* \tag{7.1}$$

$$\boldsymbol{\varepsilon}^* = \frac{1}{1-\nu}\left(\mathbf{e}\cdot\mathbf{T}_0\cdot\mathbf{e} + \frac{\nu}{1+\nu}\mathbf{g}\mathrm{tr}\mathbf{T}_0\right)$$

$$\mathbf{K}_0 = h^{-1}\mathbf{K},\quad \mathbf{M}_0 = \frac{12(1-\nu^2)}{Eh^2}\mathbf{M},\quad \mathbf{T}_0 = \frac{1-\nu^2}{Eh}\mathbf{T} \tag{7.2}$$

From equality (7.1), we obtain the constitutive relations of the conjugate plate

$$\boldsymbol{\varepsilon}^* = -(1-\nu)\left(\mathbf{e}\cdot\mathbf{T}_0\cdot\mathbf{e} - \frac{\nu}{1-\nu}\mathbf{g}\mathrm{tr}\mathbf{T}_0\right) \tag{7.3}$$

Comparing equalities (7.2) and (7.3) and taking account of the fact that the factors in front of the brackets in these formulae can be dropped at the cost of the normalizing the functions $\boldsymbol{\lambda}$ and $\mathbf{f}$, we see that, for the transition to the conjugate plate, it is necessary to replace ν by $-\nu$. The equivalence of the plane problem and the bending problem (with ν replaced by $-\nu$ when there are no disclinations and mass forces has been noted earlier.[10]

We will now consider another pair of problems concerning the equilibrium of a plate. The first is the problem of the bending of a rigidly fastened plate loaded with a transverse pressure $p(x_1, x_2)$ and distributed moments with a density $\boldsymbol{\mu}(x_1, x_2)$. The system of equations of the first problem includes the second equation of (1.1), the second relation of (1.2) and the second relation of (1.3). The second problem involves the determination of the plane stress state due to distributed boundary dislocations with a density $\boldsymbol{\alpha}$ and distributed wedge disclinations with a density τ. There are no external forces in the second case. The first equilibrium equation of (1.1), when $\mathbf{f} = 0$, can be satisfied by expressing the stress tensor in terms of the Airy stress function F.[8] The incompatibility equation (5.15) and the constitutive relation of the plane theory of elasticity have to be added to this relation. It is well known[8] that the condition that the edge of the plate is unloaded in the plane problem means that the Airy function and its normal derivative are equal to zero at each point of the boundary. The formulations of the second pair of boundary value problems for an elastic plate are shown in Table 2.

$\nabla\cdot(\nabla\cdot\mathbf{M}) + \nabla\cdot\mathbf{e}\cdot\boldsymbol{\mu} + p = 0$	$\nabla\cdot(\nabla\cdot\boldsymbol{\varepsilon}^*) + \nabla\cdot\mathbf{e}\cdot\boldsymbol{\alpha} + \tau = 0$
$\mathbf{K}^* = \mathbf{e}\cdot\nabla\nabla w\cdot\mathbf{e}$	$\mathbf{T} = \mathbf{e}\cdot\nabla\nabla F\cdot\mathbf{e}$
$\mathbf{M} = \Phi(\mathbf{K}^*)$	$\boldsymbol{\varepsilon}^* = \Phi_1(\mathbf{T})$
$w\vert_{\partial\sigma} = 0,\quad \left.\frac{\partial w}{\partial n}\right\vert_{\partial\sigma} = 0$	$F\vert_{\partial\sigma} = 0,\quad \left.\frac{\partial F}{\partial n}\right\vert_{\partial\sigma} = 0$

Subject to the condition that $\boldsymbol{\mu} = \boldsymbol{\alpha}$, $p = \tau$, $\Phi = \Phi_1$, the columns of Table 2 are mutually replaceable with the following replacements:

$$\mathbf{M} \rightleftarrows \boldsymbol{\varepsilon}^*,\quad \mathbf{K}^* \rightleftarrows \mathbf{T},\quad p \rightleftarrows \tau,\quad \boldsymbol{\mu} \rightleftarrows \boldsymbol{\alpha},\quad w \rightleftarrows F$$

This means that the problem of the plane stress state of a plate with distributed defects is mathematically equivalent to the problem of the bending of a plate with loads distributed over its surface. At the same time, in the case of an isotropic material, the solution of one problem is obtained from the solution of the other by replacing ν by $-\nu$.

The dislocation and disclination densities, as well as the external load densities, can be considered as generalized functions. This enables us to consider the case of concentrated defects. It is clear from this that the duality of the boundary-value problems for the equilibrium of a plate also holds in the case of concentrated dislocations and disclinations. For example, the problem of a concentrated wedge disclination, located at point A of a plane domain with a boundary free from external loads, is mathematically identical to the problem of the bending of a plate with a clamped boundary $\partial\sigma$ with a concentrated transverse force applied at the same point A. The problem of the bending of a plate with a free edge, due to a concentrated twisting disclination located at point A, is mathematically equivalent to the plane problem of a plate of the same shape with a clamped edge, loaded with a concentrated force at point A, which acts in the plane of the plate.

The analogy between the problems of the bending of a plate and the plane problem in the theory of elasticity which has been described above can also be extended to inhomogeneous boundary conditions. We do not discuss thus here since the case of inhomogeneous boundary conditions, but homogeneous equilibrium and compatibility equations, has been considered earlier.[10]

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